

GOLD RATIO AND A TRIGONOMETRIC IDENTITY

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ABSTRACT. We give two proofs of the identity

$$\sqrt{\frac{\cos \frac{2\pi}{5}}{\cos \frac{\pi}{5}}} + \sqrt{\frac{\cos \frac{\pi}{5}}{\cos \frac{2\pi}{5}}} = \sqrt{5},$$

using and not using the gold ratio.

1. INTRODUCTION

The *gold ratio* $\varphi = 1.61803398\dots$ is one of the most known and astonishing number in mathematics and culture. Its very interesting story is described in, e.g., [1]. It is defined as the positive root of equation

$$(1) \quad x^2 - x - 1 = 0,$$

such that

$$(2) \quad \varphi = \frac{1 + \sqrt{5}}{2}.$$

The following formulas for the gold ratio are very known:

$$(3) \quad \varphi = 2 \cos \frac{\pi}{5} = \frac{1}{2} \csc \frac{\pi}{10} = 1 + 2 \sin \frac{\pi}{10}, \dots$$

In this note we give two proofs of a trigonometric identity

$$(4) \quad \sqrt{\frac{\cos \frac{2\pi}{5}}{\cos \frac{\pi}{5}}} + \sqrt{\frac{\cos \frac{\pi}{5}}{\cos \frac{2\pi}{5}}} = \sqrt{5}$$

using and not using the gold ratio. Note that identity (4) has a Ramanujan type structure (cf. [2, p. 326]).

2. FIRST PROOF OF IDENTITY 4

Multiplying (1) on x^{n-1} , $n \geq 0$, we have

$$(5) \quad x^{n+1} = x^n + x^{n-1}.$$

Using consequently (5) for $n = 2$ and $n = 1$, we find

$$x^3 = x^2 + x = 2x + 1.$$

Thus

$$x(x^2 - 2) = 1, \quad x = \frac{1}{x^2 - 1}, \quad x^2 = \frac{x}{x^2 - 1}, \quad x = \sqrt{\frac{x}{x^2 - 2}}$$

and

$$(6) \quad x = \sqrt{\frac{\frac{x}{2}}{\frac{x^2}{2} - 1}}.$$

According to first formula (3), $\varphi = x_1 = 2 \cos \frac{\pi}{5}$ satisfies (6), and we find

$$(7) \quad \varphi = \sqrt{\frac{\cos \frac{\pi}{5}}{2 \cos^2 \frac{\pi}{5} - 1}} = \sqrt{\frac{\cos \frac{\pi}{5}}{\cos \frac{2\pi}{5}}}.$$

Now in order to prove (4) it is sufficient to verify the equality

$$(8) \quad \varphi + \frac{1}{\varphi} = \sqrt{5}.$$

Indeed, the equation

$$(9) \quad y^2 - \sqrt{5}y + 1 = 0$$

has the roots $y_1 = \frac{\sqrt{5}+1}{2} = \varphi$ and $y_2 = \frac{\sqrt{5}-1}{2} = \frac{2}{\sqrt{5}+1} = \frac{1}{\varphi}$, and, by the Vieta theorem, we obtain (8). ■

3. SECOND PROOF OF IDENTITY 4

In the second proof we follow to the scheme of our work [3]. Note that

$$(10) \quad \cos \alpha \cos 2\alpha = \frac{\sin 2\alpha \cos 2\alpha}{2 \sin \alpha} = \frac{\sin 4\alpha}{4 \sin \alpha}.$$

Substituting in (10) $\alpha = \pi/5$, we find

$$(11) \quad \cos \frac{\pi}{5} \cos \frac{2\pi}{5} = \frac{\sin \frac{4\pi}{5}}{4 \sin \frac{\pi}{5}} = 1/4.$$

On the other hand, we have

$$(12) \quad \begin{aligned} \cos \frac{2\pi}{5} - \cos \frac{\pi}{5} &= \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = \frac{2 \sin \frac{\pi}{5}}{2 \sin \frac{\pi}{5}} (\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5}) = \\ &= \frac{1}{2 \sin \frac{\pi}{5}} (\sin \frac{3\pi}{5} - \sin \frac{\pi}{5} + \sin \pi - \sin \frac{3\pi}{5}) = -\frac{1}{2}. \end{aligned}$$

Putting

$$(13) \quad x = -\cos \frac{\pi}{5}, \quad y = \cos \frac{2\pi}{5},$$

we have

$$(14) \quad xy = -\frac{1}{4}, \quad x + y = -\frac{1}{2}.$$

Denoting

$$(15) \quad A = \sqrt{-\frac{x}{y}} + \sqrt{-\frac{y}{x}},$$

we have

$$-\frac{x}{y} - \frac{y}{x} = A^2,$$

or

$$2 - A^2 = \frac{(x+y)^2 - 2xy}{xy} = -4\left(\frac{1}{4} + \frac{1}{2}\right) = -3.$$

Thus, since $A > 0$,

$$A = \sqrt{5}.$$

Therefore, from (13) and (15) we find (4). ■

REFERENCES

- [1] 1. M. Livio. The Golden Ratio: the Story of Phi, the World's Most Astonishing Number. New York: Broadway Books, 2002.
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